Decision Optimization And Pricing Strategy On Taxi-Passenger Matching Process

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***Abstract*—The taxi-passenger matching process has been studied for a long time, but still remains challenging. In this work, we try to investigate the optimal decision of taxis and the optimal pricing strategy of the firm who owns the system. To be specific, we propose a method to compute the Laplacian transform of the conditional response time of taxi driver when the matching time and inter-arrival time of customers are generally distributed, which will help to calculate the utility function of taxi drivers. We also characterize the value function of pricing strategy when the system is Markovian, which implies an optimal pricing strategy for taxi driver when passenger's arrival rate is high. Numerical simulation shows that our proposed method is correct and effective.**

***Index Terms*—taxi-passenger matching, double-ended queueing system, optimal decision, optimal pricing**

I. Introduction

Taxi-passenger matching process has received the attention from researchers long time ago. Ever since the 1950s, numerous work has been devoted to study the Taxi-passenger matching process ([1], [2], [3], [4], [5], [6], [7]), by modelling such process as a double-ended queueing system.

As is shown in Fig. 1, in a double-ended queueing system, passengers and taxis will come into the system from two sides and match with each other. A passenger and a taxi will leave the system after being matched with each other. The matching time, inter-arrival time of passengers and inter-arrival time of taxis all follow a certain distribution. The matching will take place once both sides of the system is non-empty. The double-ended queueing system can be generally applied to any process with a supply side and a demand side, not only the taxi-passenger matching process.

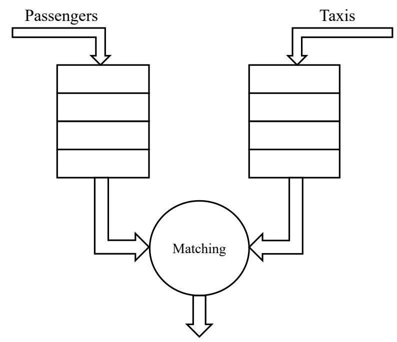


Fig. 1. A illustration of taxi-passenger matching process with double-ended queueing system. Passengers and taxis will come into the system from two sides and match with each other. A passenger and a taxi will leave the system after being matched with each other. The matching time, inter-arrival time of passengers and inter-arrival time of taxis all follow a certain distribution. The matching will take place once both sides of the system is non-empty.

With the development of shared transportation, taxi-passenger matching process has increasingly affected daily economics. However, few works have devoted to the economic side of taxi-passenger matching process, e.g. at which time to join the queueing system will maximize one taxi driver's profit, how much should the firm who owns the system charge each taxi driver to maximize the firm's income. In this work, we try to model the taxi driver's decision process and investigate the pricing strategy of the system owner to maximize the profit of both taxi drivers and owner.

Our main contributions can be concluded as follows:

- We proposed a method for computing the Laplacian transform of the conditional response time in a double-ended queueing system. The inter-arrival time of passengers and matching time in this system can be generally distributed.
- We investigated the value function of the pricing policy when the system is Markovian, which implies an optimal pricing strategy for the firm who owns the system when the passenger side is non-empty and the passenger's arrival rate is high.
- We implemented a simulation system for double-ended queueing system and tested the correctness and effectiveness of the proposed method.

II. Related Work

Numerous studies have been conducted on double-ended queueing system, since the system was firstly proposed by Kendall in [1] to model the matching process of taxis and passengers. The system consists of two sides: a stream of taxis arriving in a Poisson process with rate λ_1, and a stream of passengers arriving in a Poisson process with rate λ_2. Matching takes on when both sides of the queue are non-empty, and the matching process takes no time, which means there is a queue in a system only if one side of the system is empty. When the capacity for taxis and passengers are both infinity, it is provable that the double-ended queue doesn't have steady-state probability under constant arrival rates as shown in [1]. When the capacity for taxis and passengers are limited and the matching time is zero, Mendoza et al. [5] derived the steady-state probability of the system:

$$\pi_m = \begin{cases} \rho^{m+N}\frac{1-\rho}{1-\rho^{M+N+1}} & \rho \neq 1 \\ \frac{1}{M+N+1} & \rho = 1 \end{cases} \tag{1}$$

where π_m is the probability when m taxis are waiting in the queue (when $m < 0$, then $-m$ passengers are waiting in the queue), M is the capacity upper bound on taxis, N is the capacity upper bound on passengers and $\rho = \frac{\lambda_1}{\lambda_2}$. This result is obtained by solving the balance equation of the induced Markov chain.

Kasgyap [2] investigated a double-ended queueing system with limited waiting space and zero matching time, also he assumed the arrival process of taxis are general. Using the results in [4], Kasgyap derived the Laplacian transform of time-dependent generating function $f(x, \alpha, t)$, which is defined as:

$$f(x, \alpha, t) = \sum_{n=-N}^{M} \alpha^n p_n(x, t) \tag{2}$$

where $p_n(x, t)dx$ denotes the probability that at time t, n taxis are waiting in the queue, and the time elapsed since the last arrival of a taxi lies in $(x, x + dx)$. Bhat et al. [3] further considered the control problem on such a system by calling extra taxis when the number of passengers waiting in queue is larger than a threshold. Using renewal-reward theorem, he obtained the transient and steady-state probability of such process and the optimal control variable.

However, the assumption that the matching time is zero tends to be inconsistent with the real-world situation. Several studies have been conducted to tackle this problem. Kim et al. [6] designed a simulation method to investigate double-ended queueing system with non-zero matching time so that the model can be closer to actual situaltion. Shi et al. [7] used a matrix-analytic method to derive the steady-state probability when the matching time is non-zero. But their way in [7] to compute the joint steady-state probability involves solving a matrix-quadratic equation, which is complex and time-consuming. And they also assumed the distribution of the matching time and the distribution of inter-arrival time of taxis and passengers to be exponential, which may fail to model some situation in the real world. For example, Gong et al. [8] showed that the time of passengers walking from an airport terminal to the taxi pool of an airport satisfies inverse Gaussian distribution, under this circumstance, the inter-arrival time of passengers cannot be simply modeled as exponentially distributed.

In terms of decision modeling and pricing strategy, Chen et al. [9] proposed a state-dependent pricing strategy for a M/M/1 queueing system, and proved the optimality of such pricing strategy. Ying et al. [10] studied the the optimal control for double-ended queueing system, which involves the decision model of taxi drivers and the optimal tax policy of the government. However, due to the complexity of the system, they only focused on numerical analysis and also assumed the matching time to be zero. To the best of our knowledge, no previous work have investigated the topic of decision modeling and pricing strategy in a double-ended queueing system with non-zero matching time.

III. Decision Modeling Of Taxi Drivers

In this section, we assume the taxi drivers could decide whether to join the queue when they enter the double-ended queueing system, and try to model the decision of taxi drivers. We assume the matching time to be i.i.d random variables following general distribution D_S with its p.d.f being $f_S(x)$ and c.d.f being $F_S(x)$, we also assume the inter-arrival time of passengers to be i.i.d random variables following general distribution D_A with its p.d.f being $f_A(x)$ and c.d.f being $F_A(x)$. The matching process is assumed to have a FCFS scheduling.

A. Conditional Response Time Analysis

When a taxi arrives and finds that $M - 1$ other taxi drivers and N passengers are waiting in the queue, we try to derive the response time of that taxi under such condition. Here we let $T_{M,N}$ denote the response time of the taxi driver when there are $M - 1$ other taxis ahead of him and N passengers are currently waiting in queue. Consider the situation when the passenger side of the system is empty, we have the following proposition:

Proposition 1. let S_i be i.i.d random variables following D_S for $i = 1, 2, \ldots, M$, A_i be i.i.d random variables following D_A for $i = 1, 2, \ldots, M - 1$ and A_e be the excess time of the arrival of next passenger. We have:

$$T_{M,0} = \max_{j\in[0,M-1]} (\sum_{i=1}^{j} A_i + \sum_{i=j+1}^{M-1} S_i) + S_M + A_e \tag{3}$$

Proof. We prove this by mathematical induction. When $M = 1$, this taxi needs to wait for the first passenger to arrive and finish the matching procedure. So clearly $T_{1,0} = S_1 + A_e$. Now suppose the proposition holds for all $1 \leq j < M$, Then the taxi will start its matching process only when the following two events have all happened

- The taxi directly in front of it has left the system.
- M passengers have entered the system

Therefore we have:

$$\begin{aligned} T_{M,0} &= \max(T_{M-1,0}, \sum_{i=1}^{M-1} A_i + A_e) + S_M \\ &= \max(\max_{j\in[0,M-2]} (\sum_{i=1}^{j} A_i + \sum_{i=j+1}^{M-2} S_i) + S_{M-1}, \sum_{i=1}^{M-1} A_i) \\ &+ S_M + A_e \\ &= \max_{j\in[0,M-1]} (\sum_{i=1}^{j} A_i + \sum_{i=j+1}^{M-1} S_i) + S_M + A_e \end{aligned}$$

So the proposition holds for every integer $M \geq 1$. $\square$

The result of Proposition 1 can be generalized to the situation where the passenger side of the system is non-empty, but then we need to consider the excess time of a matching process, denoted by S_e.

Proposition 2. let S_i be i.i.d random variables following D_S for $i = 1, 2, \dots, M-1$, S_e be the excess time of the completion of a matching, A_i be i.i.d random variables following D_A for $i = 1, 2, \dots, M-N-1$ and A_e be the excess time of the arrival of next passenger. We have:

$$T_{1,N} = S_1, \forall N > 0 \tag{4}$$

$$T_{M,N} = S_e + \sum_{i=1}^{M-1} S_i, \forall N > 0, 1 < M \leq N \tag{5}$$

$$\begin{aligned} T_{M,N} = &\max[S_e + \sum_{i=1}^{M-2} S_i, \\ &A_e + \max_{j\in[0,M-N-1]}(\sum_{i=1}^{j} A_i + \sum_{i=j+1}^{M-N-1} S_{N+i-1})] \\ &+ S_{M-1}, \forall N > 0, M > N \end{aligned} \tag{6}$$

Proof. When $M \leq N$, the newly arrived taxi just need to wait the taxis ahead of him to match their own passengers, then take its passenger to leave the system, and when $M > 1$, the time for the taxi in the front of the queue to finish matching is S_e. So (8) and (5) clearly holds. When $M > N$, we can also use mathematical induction to get the results. Notice that for the newly arrived taxi to get served, two events must all have happened:

- The taxi directly in front of it has left the system.
- $M - N$ passengers have entered the system

So we have:

$$\begin{aligned} T_{M,N} &= \max(T_{M-1,N}, \sum_{i=1}^{M-N-1} A_i + A_e) + S_{M-1} \\ &= \max(\max[S_e + \sum_{i=1}^{M-3} S_i, \\ &A_e + \max_{j\in[0,M-N-2]}(\sum_{i=1}^{j} A_i + \sum_{i=j+1}^{M-N-2} S_{N+i-1})] \\ &+ S_{M-2}, \sum_{i=1}^{M-N-1} A_i + A_e) + S_{M-1} \\ &= \max[S_e + \sum_{i=1}^{M-2} S_i, \\ &A_e + \max_{j\in[0,M-N-1]}(\sum_{i=1}^{j} A_i + \sum_{i=j+1}^{M-N-1} S_{N+i-1})] \\ &+ S_{M-1} \end{aligned}$$

So (6) holds for all $M > N$. □

With proposition 1 and proposition 2, we can calculate the conditional mean response time.

Corollary 1.

$$\mathbf{E}[T_{M,0}] = \sum_{i=0}^{M-1} \pi_i^{(M)} \cdot [i \cdot \mathbf{E}[A] + (M-i) \cdot \mathbf{E}[S]] + \frac{\mathbf{E}[A^2]}{2\mathbf{E}[A]} \tag{7}$$

$$\mathbf{E}[T_{M,N} \Big| 1 < M \leq N, N > 0] = (M-1) \cdot \mathbf{E}[S] + \frac{\mathbf{E}[S^2]}{2\mathbf{E}[S]} \tag{8}$$

$$\begin{aligned} &\mathbf{E}[T_{M,N} \Big| M > N, N > 0] \\ &= \pi_{0,0}^{(M,N)} \cdot [(M-1) \cdot \mathbf{E}[S] + \frac{\mathbf{E}[S^2]}{2\mathbf{E}[S]}] \\ &+ \sum_{i=0}^{M-N-1} \pi_{1,i}^{(M,N)} \cdot [i \cdot \mathbf{E}[A] + (M-N-i) \cdot \mathbf{E}[S] + \frac{\mathbf{E}[A^2]}{2\mathbf{E}[A]}] \end{aligned} \tag{9}$$

where:

$$\pi_i^{(M)} = \Pr[\operatorname*{argmax}_{j\in[0,M-1]}(\sum_{k=1}^{j} A_k + \sum_{k=j+1}^{M-1} S_k) = i]$$

$$\begin{aligned} \pi_{0,0}^{(M,N)} = &\Pr[S_e + \sum_{i=1}^{M-2} S_i > \\ &A_e + \max_{j\in[0,M-N-1]}(\sum_{i=1}^{j} A_i + \sum_{i=j+1}^{M-N-1} S_{N+i-1})] \end{aligned}$$

$$\begin{aligned} \pi_{1,i}^{(M,N)} = &\Pr[S_e + \sum_{k=1}^{M-2} S_k \leq \\ &A_e + \max_{j\in[0,M-N-1]}(\sum_{k=1}^{j} A_k + \sum_{k=j+1}^{M-N-1} S_{N+k-1}), \\ &\operatorname*{argmax}_{j\in[0,M-N-1]}(\sum_{k=1}^{j} A_k + \sum_{k=j+1}^{M-N-1} S_{N+k-1}) = i] \end{aligned}$$

(7), (8) and (9) can be derived from (3), (5) and (6) using the law of total expectation. And we have used the results derived in [11]:

$$\mathbf{E}[A_e] = \frac{\mathbf{E}[A^2]}{2\mathbf{E}[A]}$$

$$\mathbf{E}[S_e] = \frac{\mathbf{E}[S^2]}{2\mathbf{E}[S]}$$

Now we show that $\pi_i^{(M)}$ ($i = 0, 1, \dots, M-1$), $\pi_{0,0}^{(M,N)}$ and $\pi_{1,i}^{(M,N)}$ ($i = 0, 1, \dots, M-N-1$) can be numerically computed when given $F_S(x)$, $f_S(x)$, $F_A(x)$ and $f_A(x)$.

Proposition 3. Given $F_S(x)$, $f_S(x)$, $F_A(x)$ and $f_A(x)$, $\pi_i^{(M)}$, $\pi_{0,0}^{(M,N)}$ and $\pi_{1,i}^{(M,N)}$ are numerically computable.

Proof. From the definition of $\pi_i^{(M)}$ we can see that:

$$\begin{aligned}\pi_i^{(M)} &= \Pr[\underset{j\in[0,M-1]}{\operatorname{argmax}} (\sum_{k=1}^{j} A_k + \sum_{k=j+1}^{M-1} S_k) = i] \\ &= \Pr[\bigcap_{j=0}^{M-1} \{\sum_{k=1}^{j} A_k + \sum_{k=j+1}^{M-1} S_k \le \sum_{k=1}^{i} A_k + \sum_{k=i+1}^{M-1} S_k\}] \\ &= \Pr[\bigcap_{j=0}^{i-1} \{\sum_{k=j+1}^{i} A_k \ge \sum_{k=j+1}^{i} S_k\}, \\ &\quad \bigcap_{j=i+1}^{M-1} \{\sum_{k=i+1}^{j} A_k \le \sum_{k=i+1}^{j} S_k\}] \\ &= \Pr[\bigcap_{j=0}^{i-1} \{\sum_{k=j+1}^{i} A_k \ge \sum_{k=j+1}^{i} S_k\}] \\ &\quad \times \Pr[\bigcap_{j=i+1}^{M-1} \{\sum_{k=i+1}^{j} A_k \le \sum_{k=i+1}^{j} S_k\}]\end{aligned}$$

Let $X_k = A_k - S_k$, then $X_1, X_2, \ldots, X_{M-1}$ are i.i.d random variables, the p.d.f of X_k, $f_X(x)$ can be calculated as:

$$f_X(x) = \int_{-\infty}^{+\infty} f_A(x-t) f_S(-t) dt$$

And the tail distribution of X_k is:

$$\overline{F_X}(x) = \int_{x}^{+\infty} f_X(t) dt$$

Then we have:

$$\begin{aligned}&\Pr[\bigcap_{j=0}^{i-1} \{\sum_{k=j+1}^{i} A_k \ge \sum_{k=j+1}^{i} S_k\}] \\ &= \Pr[\bigcap_{j=1}^{i} \{\sum_{k=1}^{j} X_k \ge 0\}] \\ &= \prod_{j=1}^{i} \Pr[\sum_{k=1}^{j} X_k \ge 0 \Bigm| \bigcap_{l<j} \{\sum_{k=1}^{l} X_k \ge 0\}]\end{aligned}$$

We recursively calculate the conditional probability as follows:

$$\begin{aligned}&\Pr[\sum_{k=1}^{j} X_k \ge x \Bigm| \bigcap_{l<j} \{\sum_{k=1}^{l} X_k \ge 0\}] \\ &= \int_{0}^{+\infty} \overline{F_X}(x-t) \Pr[\sum_{k=1}^{j-1} X_k \in (t, t+dt) \Bigm| \bigcap_{l<j} \{\sum_{k=1}^{l} X_k \ge 0\}]\end{aligned}$$

Similarly, we can calculate:

$$\Pr[\bigcap_{j=i+1}^{M-1} \{\sum_{k=i+1}^{j} A_k \le \sum_{k=i+1}^{j} S_k\}]$$

So $\pi_i^{(M)}$ is numerically computable.

For $p_{0,0}$, we can derive that:

$$\begin{aligned}\pi_{0,0}^{(M,N)} &= \Pr[\bigcap_{j\in[0,M-N-1]} \{\sum_{k=1}^{j} X_k < S_e - A_e + \sum_{k=1}^{N-1} S_k\}] \\ &= \int_{-\infty}^{+\infty} \Pr[\bigcap_{j=0}^{M-N-1} \{\sum_{k=1}^{j} X_k < t\}] \\ &\quad \times \Pr[S_e - A_e + \sum_{k=1}^{N-1} S_k \in (t, t+dt)]\end{aligned}$$

The distribution of A_e and S_e can be obtained using results in [11], and the distribution of $\sum_{k=1}^{N-1} S_k$ can be obtained by recursively calculating the convolution. So $\pi_{0,0}^{(M,N)}$ is also numerically computable.

For $\pi_{1,i}^{(M,N)}$, we can derive that:

$$\begin{aligned}\pi_{1,i}^{(M,N)} &= \Pr[\underset{j\in[0,M-N-1]}{\operatorname{argmax}} (\sum_{k=1}^{j} X_k) = i, \\ &\quad \sum_{k=1}^{i} X_k \ge S_e - A_e + \sum_{k=1}^{N-1} S_k] \\ &= \int_{-\infty}^{+\infty} \Pr[\underset{j\in[0,M-N-1]}{\operatorname{argmax}} (\sum_{k=1}^{j} X_k) = i, \sum_{k=1}^{i} X_k \ge t] \\ &\quad \times \Pr[S_e - A_e + \sum_{k=1}^{N-1} S_k \in (t, t+dt)]\end{aligned}$$

Then $\pi_{1,i}^{(M,N)}$ can be calculated the same way as $\pi_i^{(M)}$, except for an extra condition. So $\pi_{1,i}^{(M,N)}$ is also numerically computable. $\square$

B. Optimal Decision Of Taxi Drivers

Suppose the taxi driver could receive R reward when immediately finish matching, and there is an outside non-stochastic reward v when the taxi driver choose not to join the queue. The firm who owns the double-ended queueing system would choose to charge a price p for the taxi choosing to join the queue. We let $V_{M,N}$ denote the estimated reward of one taxi driver who arrives in the system with $M-1$ other taxis ahead and N passengers waiting in the queue. Let $s_{M,N}$ denote the decision of the taxi driver to join the queue ($s_{M,N} = 1$) or directly leave the system ($s_{M,N} = 0$). Here we use the same estimation function as in [9]:

$$V_{M,N} = \begin{cases} \mathbf{E}\Big[e^{-\gamma\tau} R - p - c\int_0^{\tau} e^{-\gamma t} dt\Big] & , s_{M,N} = 1 \\ v & , s_{M,N} = 0 \end{cases} \tag{10}$$

where $c \ge 0$ is the unit cost of delay and $0 \le \gamma \le 1$ is the discount rate, $\tau = T_{M,N}$ is the response time. The parameter c and γ allows a general and realistic modeling of taxi drivers preference, when $\gamma = 0$, the reward to join the queue is just a linear function of response time and when $c = 0$ we obtain a situation where the response time only acts as a discounting factor. To maximize the net reward, the taxi driver will join the queue if and only if:

$$\mathbf{E}\Big[e^{-\gamma\tau}R - p - c\int_0^{\tau} e^{-\gamma t}dt\Big] \geq v$$

Proposition 4. Given $f_S(x)$ and $f_A(x)$, the estimation function shown in (10) can be numerically computed.

Proof. Using the linearity of expectation, we have:

$$\begin{aligned}&\mathbf{E}\Big[e^{-\gamma\tau}R - p - c\int_0^{\tau} e^{-\gamma t}dt\Big] \\ &= (R+\frac{c}{\gamma})\widetilde{T_{M,N}}(\gamma) - p - \frac{c}{\gamma}\end{aligned}$$

where $\widetilde{T_{M,N}}(\cdot)$ is the Laplacian transform of $T_{M,N}$. Let $\widetilde{A}(\cdot)$ denote the Laplacian transform of random variables following D_A, and $\widetilde{S}(\cdot)$ denote the Laplacian transform of random variables following D_S. We can compute $\widetilde{A}(\cdot)$ and $\widetilde{S}(\cdot)$ as follows:

$$\widetilde{A}(x) = \int_0^{+\infty} e^{-xt}f_A(t)dt$$
$$\widetilde{S}(x) = \int_0^{+\infty} e^{-xt}f_S(t)dt$$

When $N=0$, from (3) we have:

$$\widetilde{T_{M,0}}(\gamma) = \frac{1-\widetilde{A}(\gamma)}{\gamma\mathbf{E}[A]}\sum_{i=0}^{M-1}\pi_i^{(M)}\cdot\widetilde{A}^i(\gamma)\cdot\widetilde{S}^{M-i}(\gamma)$$

when $N>0$ and $M\leq N$, from (5) we have:

$$\widetilde{T_{M,N}}(\gamma) = \frac{1-\widetilde{S}(\gamma)}{\gamma\mathbf{E}[S]}\cdot\widetilde{S}^{M-1}(\gamma)$$

when $M>N>0$, from (6) we have:

$$\begin{aligned}\widetilde{T_{M,N}}(\gamma) &= \pi_{0,0}^{(M,N)}\cdot\frac{1-\widetilde{S}(\gamma)}{\gamma\mathbf{E}[S]}\cdot\widetilde{S}^{M-1}(\gamma) \\ &+ \frac{1-\widetilde{A}(\gamma)}{\gamma\mathbf{E}[A]}\sum_{i=0}^{M-N-1}\pi_{1,i}^{(M,N)}\cdot\widetilde{A}^i(\gamma)\cdot\widetilde{S}^{M-N-i}(\gamma)\end{aligned}$$

Here we directly use the results from [11]:

$$\widetilde{A_e}(\gamma) = \frac{1-\widetilde{A}(\gamma)}{\gamma\mathbf{E}[A]}$$
$$\widetilde{S_e}(\gamma) = \frac{1-\widetilde{S}(\gamma)}{\gamma\mathbf{E}[S]}$$

where $\widetilde{A_e}(\cdot)$ and $\widetilde{S_e}(\cdot)$ are the Laplacian transform of A_e and S_e respectively. So, we can numerically compute the Laplacian transform of $T_{M,N}$, which is the key to compute the estimation function. So (10) can be computed numerically. □

With the above proposition, a taxi driver can then make the optimal decision upon arriving at the system to maximize the estimation function shown in (10).

IV. Pricing Strategy Of The Firm

In this section, we assume the firm who owns the double-ended queueing system can observe the current state of the whole system, i.e. the number of taxis and passengers waiting in the queue. And the firm will charge the taxi who decides to join the queue at time t a price p_t. Let $M(t)$ denote the total number of taxis who actually decide to join the queue by time t. Inspired by Chen et al. [9], we model the pricing strategy of the firm as to choose a set of prices $p=\{p_t, t\leq 0\}$ in order to maximize the expected discounted revenue:

$$\mathbf{E}\Big[\int_0^{+\infty} e^{-\gamma t}p_t dA(t)\Big] \tag{11}$$

The model will get quite complex when the distribution of some variables in the model is general. To make the whole decision process Markovian, we assume the inter-arrival time between two taxis follows exponential distribution with the rate being λ_1, the inter-arrival time between two passengers follows exponential distribution with the rate being λ_2, and the time for a matching process is a random variable following exponential distribution with parameter μ. Here we also define a state of the system as a tuple $s=(M,N)$, indicating that there are M taxis and N passengers waiting in the system. Let $\hat{s}(p_t)$ denote the set of states at which the optimal decision for a arriving taxi driver is to join the queue when the price is p_t. Then the firm's objective function can be restated as:

$$\mathbf{E}\Big[\int_0^{+\infty} e^{-\gamma t}\lambda_1 p_t\mathbf{I}_{\{s(t)\in\hat{s}(p_t)\}}dt\Big] \tag{12}$$

where $\mathbf{I}_{\{s(t)\in\hat{s}(p_t)\}}$ is a indicator which equals to 1 iff the state at time t belongs to $\hat{s}(p_t)$.

Let $V(M,N)$ denote the optimal expected profit when the initial state is $s=(M,N)$.Since now the whole process is Markovian, we have:

$$\begin{aligned}&V(M,N) \\ &= \frac{1}{\mu+\lambda_1+\lambda_2+\gamma}\max_{p\geq 0}\Big\{\mu[\mathbf{I}_{\{M>0,N>0\}}V(M-1,N-1) \\ &\quad+\mathbf{I}_{\{M=0,N>0\}}V(0,N)+\mathbf{I}_{\{M>0,N=0\}}V(M,0) \\ &+\mathbf{I}_{\{M=0,N=0\}}V(0,0)]+\lambda_1\mathbf{I}_{\{(M,N)\in\hat{s}(p)\}}[p+V(M+1,N)] \\ &\quad+\lambda_1\mathbf{I}_{\{(M,N)\notin\hat{s}(p)\}}V(M,N)+\lambda_2V(M,N+1)\Big\}\end{aligned}$$

Move the terms that are independent of p out of the max$(\cdot)$ operation, we have:

$$\begin{aligned}&V(M,N) \\ &= \frac{\mu}{\mu+\lambda_1+\lambda_2+\gamma}[\mathbf{I}_{\{M>0,N>0\}}V(M-1,N-1) \\ &\quad+\mathbf{I}_{\{M=0,N>0\}}V(0,N)+\mathbf{I}_{\{M>0,N=0\}}V(M,0) \\ &+\mathbf{I}_{\{M=0,N=0\}}V(0,0)]+\frac{\lambda_2}{\mu+\lambda_1+\lambda_2+\gamma}V(M,N+1) \\ &\quad+\frac{\lambda_1}{\mu+\lambda_1+\lambda_2+\gamma}\max_{p\geq 0}\Big\{\ \mathbf{I}_{\{(M,N)\notin\hat{s}(p)\}}V(M,N) \\ &\quad+\mathbf{I}_{\{(M,N)\in\hat{s}(p)\}}[p+V(M+1,N)]\Big\}\end{aligned} \tag{13}$$

We define $p_{M,N}$, which is equal to the maximum amount of price that a driver can accept when he or she sees N passengers and $M-1$ taxis in the system when getting served, as follows:

$$p_{M,N} = (R + \frac{c}{\gamma})\widetilde{T_{M,N}}(\gamma) - v - \frac{c}{\gamma} \tag{14}$$

For any price p, since state (M,N) will either belong to $\hat{s}(p)$ or not. And when $(M,N) \in \hat{s}(p)$, the way to maximize the expected reward is to maximize the price p, so inequality (13) can be modified as:

$$\begin{aligned} &V(M,N) \\ &= \frac{\mu}{\mu+\lambda_1+\lambda_2+\gamma}[\mathbf{I}_{\{M>0,N>0\}}V(M-1,N-1) \\ &\quad + \mathbf{I}_{\{M=0,N>0\}}V(0,N) + \mathbf{I}_{\{M>0,N=0\}}V(M,0) \\ &+ \mathbf{I}_{\{M=0,N=0\}}V(0,0)] + \frac{\lambda_2}{\mu+\lambda_1+\lambda_2+\gamma}V(M,N+1) + \\ &\frac{\lambda_1}{\mu+\lambda_1+\lambda_2+\gamma}\max\Big\{V(M,N), p_{M,N} + V(M+1,N)\Big\} \end{aligned} \tag{15}$$

To get the optimal pricing strategy, we first prove the following lemmas.

Lemma 1. $p_{M,N}$ is non-increasing in M.

Proof. To prove this, from (14), we only need to prove that $\widetilde{T_{M,N}}(\lambda)$ is non-increasing in M. We let the double-ended queueing system start from any steady state at time $t=0$. Let $n(t)$ denote the number of taxis which sees $M-1$ other taxis and N passengers waiting in the queue upon arriving and complete its matching, and $T_{M,N}^{(i)}$ denote the response time of i^{th} such taxi, when $N=0$, from proposition 1 we have:

$$\begin{aligned} T_{M,N}^{(i)} &= \max(T_{M-1,N}^{(i)}, \sum_{i=1}^{M-1} A_i + A_e) + S_M \\ &\geq T_{M-1,N}^{(i)} \end{aligned}$$

Then we have:

$$\begin{aligned} \mathbf{E}[e^{-\gamma T_{M,N}}] &= \lim_{t\to\infty} \frac{\sum_{i=1}^{n(t)} e^{-\lambda T_{M,N}^{(i)}}}{n(t)} \\ &\leq \lim_{t\to\infty} \frac{\sum_{i=1}^{n(t)} e^{-\lambda T_{M-1,N}^{(i)}}}{n(t)} \\ &= \mathbf{E}[e^{-\gamma T_{M-1.N}}] \end{aligned}$$

which means $\widetilde{T_{M,N}}(\lambda)$ is non-increasing in M. □

Lemma 2. For any function $f(M)$, let $g(M) = \max(p_{M,N} + f(M+1), f(M))$. If $p_{M,N} + f(M+1) - f(M)$ is non-increasing in M, then $p_{M,N} + g(M+1) - g(M)$ is non-increasing in M.

Proof. The whole proof is similar to the proof in [9]. The non-increasing assumption implies that, for any $M \geq 0$:

$$\begin{aligned} &p_{M,N} + f(M+1) - f(M) \\ &\geq p_{M+1,N} + f(M+2) - f(M+1) \end{aligned} \tag{16}$$

We need to prove:

$$\begin{aligned} \Delta \equiv &[p_{M,N} + g(M+1) - g(M)] \\ &- [p_{M+1,N} + g(M+2) - g(M+1)] \\ &\geq 0 \end{aligned}$$

Now consider the following four cases with fixed M:

1) When $p_{M+2,N} + f(M+3) - f(M+2) \geq 0$, by (16), we have:
$$p_{k,N} + f(k+1) - f(k) \geq 0$$
for $k = M, M+1$. Hence:
$$\begin{aligned} \Delta = &[p_{M+1,N} + f(M+2) - f(M+1)] \\ &- [p_{M+2,N} + f(M+3) - f(M+2)] \\ &\geq 0 \end{aligned}$$
2) When $p_{M+2,N} + f(M+3) - f(M+2) < 0 \leq p_{M+1,N} + f(M+2) - f(M+1)$, similarly, from (16), we have:
$$\Delta = p_{M+1,N} + f(M+2) - f(M+1) \geq 0$$
3) When $p_{M+1,N} + f(M+2) - f(M+1) < 0 \leq p_{M,N} + f(M+1) - f(M)$, we have:
$$\Delta = -[p_{M+1,N} + f(M+2) - f(M+1)] > 0$$
4) When $p_{M,N} + f(M+1) - f(M) < 0$, in this case, we have:
$$\begin{aligned} \Delta = &[p_{M,N} + f(M+1) - f(M)] \\ &- [p_{M+1,N} + f(M+2) - f(M+1)] \\ &\geq 0 \end{aligned}$$

□

Now we consider some of the character of the value function described in (15).

Proposition 5. When two conditions are satisfied:

1)
$$\frac{p_{M,N} - p_{M+1,N}}{p_{M-1,N-1} - p_{M,N-1}} \geq \frac{\mu}{\mu+\lambda}$$
2)
$$\frac{p_{M,N} - p_{M+1,N}}{p_{M,N+1} - p_{M+1,N+1}} \geq 1$$

Then, when $M > 0, N \geq 0$, the value function $V(M,N)$ has the following two characters:

- $V(M,N)$ is non-increasing in M,
- $p_{M,N} + V(M+1,N) - V(M,N)$ is non-increasing in M

Proof. We adopt the value iteration approach as in [12] to prove this proposition. For simplicity, we assume $\mu + \lambda_1 + \lambda_2 + \gamma = 1$. When $M > 0, N \geq 0$, we approximate V through the following recursion:

$$\begin{aligned} &V_{m+1}(M,N) \\ &= \lambda_2 V_m(M,N+1) + \mu[\mathbf{I}_{\{N>0\}}V_m(M-1,N-1) \\ &\quad + \mathbf{I}_{\{N=0\}}V_m(M,0)] \\ &\quad + \lambda_1 \max\Big\{V_m(M,N), p_{M,N} + V_m(M+1,N)\Big\} \end{aligned} \tag{17}$$

We have:

$$\lim_{m\to\infty} V_m(M,N) = V(M,N)$$

So we prove the following inequalities for all $M > 0$, $N \geq 0$ by induction on m:

$$\begin{aligned}\Delta_{1,m} \equiv&[p_{M,N} + V_m(M+1,N) - V_m(M,N)] \\ &- [p_{M,N} + V_m(M+1,N) - V_m(M,N)] \geq 0\end{aligned} \tag{18}$$

$$\Delta_{2,m} \equiv V_m(M,N) - V_m(M+1,N) \geq 0 \tag{19}$$

Let $V_0(M,N) = 0$ for all $M > 0, N \geq 0$, and $p_{0,N} \geq 0$ for all $N \geq 0$, then (18) and (19) clearly hold for $m = 0$. Next suppose that inequality (18) and (19) hold for all $m \leq i$, we need to show that they still hold for $m = i+1$.

For (18), we let:

$$\begin{aligned}f_m(M) = \mu[&\mathbf{I}_{\{N>0\}} V_m(M-1,N-1) \\ &+ \mathbf{I}_{\{N=0\}} V_m(M,0)]\end{aligned}$$

and

$$g_m(M) = \max\Big\{V_m(M,N), p_{M,N} + V_m(M+1,N)\Big\}$$

and

$$h_m(M) = V_m(M,N+1)$$

Then (17) can be rewritten as:

$$V_{m+1}(M,N) = f_m(M) + \lambda_1 g_m(M) + \lambda_2 h_m(M)$$

Therefore, (18) can be written as:

$$\begin{aligned}\Delta_{1,i+1} = &\{[(\mu+\gamma)p_{M,N} + f_i(M+1) - f_i(M)] \\ &- [(\mu+\gamma)p_{M+1,N} + f_i(M+2) - f_i(M+1)]\} \\ &+ \lambda_1\{[p_{M,N} + g_i(M+1) - g_i(M)] \\ &- [p_{M+1,N} + g_i(N+2) - g_i(M+1)]\} \\ &+ \lambda_2\{[p_{M,N} + h_i(M+1) - h_i(M)] \\ &- [p_{M+1,N} + h_i(M+2) - h_i(M+1)]\}\end{aligned}$$

By Lemma 2, the second term in the above inequality is non-negative, so we now only need to prove:

$$\begin{aligned}\Delta^{(1)} \equiv &[(\mu+\gamma)p_{M,N} + f_i(M+1) - f_i(M)] \\ &- [(\mu+\gamma)p_{M+1,N} + f_i(M+2) - f_i(M+1)] \\ &\geq 0\end{aligned}$$

and:

$$\begin{aligned}\Delta^{(2)} \equiv &[p_{M,N} + h_i(M+1) - h_i(M)] \\ &- [p_{M+1,N} + h_i(M+2) - h_i(M+1)] \geq 0\end{aligned}$$

First from the definition of h:

$$\begin{aligned}\Delta^{(2)} = &[p_{M,N} + V(M+1,N+1) - V(M,N+1)] \\ &- [p_{M+1,N} + V(M+2,N+1) - V(M+1,N+1)] \\ = &[p_{M,N+1} + V(M+1,N+1) - V(M,N+1)] \\ &- [p_{M+1,N+1} + V(M+2,N+1) - V(M+1,N+1)] \\ &p_{M,N} - p_{M+1,N} - p_{M,N+1} + p_{M+1,N+1} \\ \geq &\, p_{M,N} - p_{M+1,N} - p_{M,N+1} + p_{M+1,N+1} \geq 0\end{aligned}$$

The last inequality comes from condition 2). When $M > 0$, $N > 0$ from the definition of f, we have:

$$\begin{aligned}&\Delta^{(1)} \\ &= [(\mu+\gamma)p_{M,N} + \mu(V_i(M,N-1) - V_i(M-1,N-1))] \\ &- [(\mu+\gamma)p_{M+1,N} + \mu(V_i(M+1,N-1) - V_i(M,N-1))] \\ &= \mu\{[p_{M-1,N-1} + V_i(M,N-1) - V_i(M-1,N-1)] \\ &\quad - [p_{M,N-1} + V_i(M+1,N-1) - V_i(M,N-1)] \\ &\quad + (\mu+\gamma)(p_{M,N} - p_{M+1,N}) - \mu(p_{M-1,N-1} - p_{M,N-1}) \\ &\geq (\mu+\gamma)(p_{M,N} - p_{M+1,N}) - \mu(p_{M-1,N-1} - p_{M,N-1}) \geq 0\end{aligned}$$

The last inequality is from condition 1). When $N = 0$, the proof is similar.

For (19), when $N > 0$, we have:

$$\begin{aligned}\Delta_{2,i+1} = &\lambda_2[V_i(M,N+1) - V_i(M+1,N+1)] \\ &+ \mu[V_i(M-1,N-1) - V_i(M,N-1)] \\ &\lambda_1[\max\Big\{V_i(M,N), p_{M,N} + V_m(M+1,N)\Big\} \\ &- \max\Big\{V_i(M+1,N), p_{M+1,N} + V_m(M+2,N)\Big\} \\ &\geq 0\end{aligned}$$

The last inequality comes from induction hypothesis and Lemma 1. When $N = 0$, the proof is similar. □

Proposition 5 reveals the fact that when the system is non-empty, then when $p_{M,N} \leq 0$, we must have $p_{M,N} + V(M+1,N) \leq V(M,N)$, so then, to get the optimal revenue, the firm should not let this taxi join the queue by setting a very high price. Since $p_{M,N} + V(M+1,N) - V(M,N)$ is non-increasing in M when $M > 0$, then for any $N \geq 0$, we can find the maximum $M_N > 0$ such that $p_{M,N} + V(M+1,N) \geq V(M,N)$. When there are N passengers in the system, the firm should let at most M_N taxis to enter the system to get the maximum revenue.

The two conditions in Proposition 5 basically say that the number of passengers waiting in queue shouldn't affect the response time too much, the response time for a taxi should mainly composed of matching time, which means the arrival rate of passengers should relatively high to make the pricing strategy effective.

V. Experiment Results

In this work, we implemented a simulation system to simulate the taxi-passenger matching process in a double-ended queue. The system was implemented with Python 3.7[1] and Numpy[2]. The decision model and pricing strategy discussed above were tested in the simulation process. To reduce the computational complexity, $\pi_i^{(M)}$, $\pi_{0,0}^{(M,N)}$ and $\pi_{1,i}^{(M,N)}$ are estimated through Monte Carlo method.

Specifically, we set the inter-arrival time of taxis as exponentially distributed with $\frac{1}{\lambda_1} = 1.1$, the service time as exponentially distributed with $\frac{1}{\mu} = 0.7$. The reward of a taxi R is set to be 10, other parameters in (10) is set as $c = 0.1$,

[1]https://www.python.org/

[2]https://numpy.org/

TABLE I
PERFORMANCE COMPARISON

Price	$\lambda_2 = 0.5$				$\lambda_2 = 2$			
	Join ratio		**Mean profit**		**Join ratio**		**Mean profit**	
	Our model	*Random*	*Our model*	*Random*	*Our model*	*Random*	*Our model*	*Random*
$p = 2$	0.60	0.51	**2.73**	2.14	0.99	0.51	**6.24**	4.57
$p = 5$	0.58	0.50	**2.63**	2.01	0.93	0.50	**3.54**	2.97
$p = 7$	0.54	0.49	**2.11**	0.78	0.61	0.50	**2.18**	1.92
$p = 10$	0.00	0.50	**2.00**	-0.31	0.00	0.51	**2.00**	0.46

TABLE II
RELATIVE ERROR

Rate / **Factor**	$\lambda_2 = 0.5$	$\lambda_2 = 2$
$\gamma = 0.1$	1.51 %	0.71 %
$\gamma = 0.3$	1.82 %	0.89 %
$\gamma = 0.5$	1.55 %	0.84 %
$\gamma = 1$	1.48 %	0.82 %

$v = 2$. In our simulation, the system initializes with 0 taxis and 0 passengers, and the simulation stops when 1000 taxis have arrived at the system.

A. Decision Model

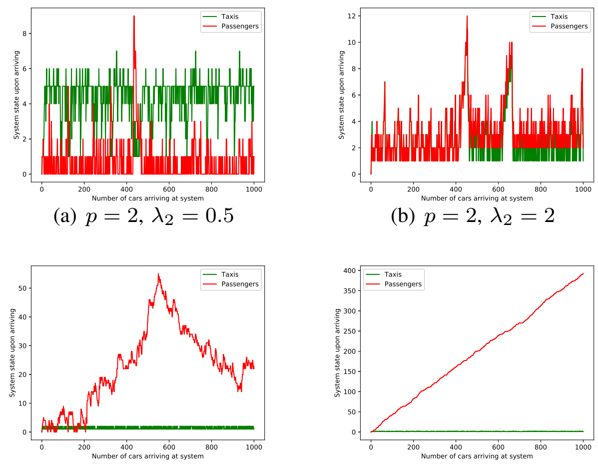
We measured the relative error of the computed Laplacian transform using the following equation:

$$\text{err} = \frac{|\sum_{i=1}^{n}[\widetilde{T}_i(\gamma) - e^{-\gamma T_i}]|}{\sum_{i=1}^{n} e^{-\gamma T_i}}$$

where T_i is observed response time of the i^{th} taxi joining the queue, $\widetilde{T}_i(\gamma)$ is the computed Laplacian transform of that taxi. Table II shows the results we measured under the situation where $\lambda_2 = 0.5$ and $\lambda_2 = 2$ with different γ. When the arrival rate of passengers is low, the computation of Laplacian transform depends more on $\pi_i^{(M)}$, $\pi_{0,0}^{(M,N)}$ and $\pi_{1,i}^{(M,N)}$, which are computed in a Monte Carlo approach, so the relative error will be higher under that situation. Yet under both situations when $\lambda_2 = 2$ and $\lambda_2 = 0.5$, the relative error is smaller than 0.02, which proves the correctness of our proposed way to model the conditional mean response time.

We also measured the mean profit of taxis following the decision process defined in (10), compared with random decision process (i.e. with probability 0.5 to join the queue). The price charged by the firm is fixed. As is illustrated in Table I, our decision model would avoid joining a queue with high price and low arrival rate of passengers, and our model outperforms the random model in terms of mean profit. This further confirms the correctness of our proposed decision modelling.

Moreover, we compared the states of the system when the price and arrival rate of passengers vary. As shown in Fig. 2,



(a) $p = 2$, $\lambda_2 = 0.5$ (b) $p = 2$, $\lambda_2 = 2$

(c) $p = 7$, $\lambda_2 = 0.5$ (d) $p = 7$, $\lambda_2 = 2$

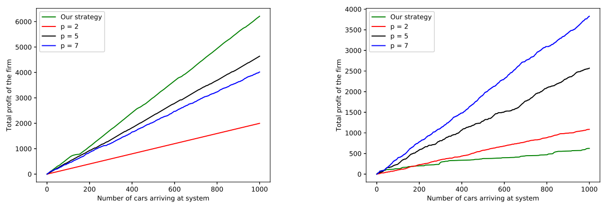
Fig. 2. Comparison between the states of the system when the price and the arrival rate of passengers vary. Green lines indicate the number of taxis and red lines indicate the number of passengers in the system. Data was collected each time a new taxi arrived at the system.

when price is relatively low, it's quite hard to form a large scale congestion on passengers, taxis will slightly congest when the arrival rate of passengers is low; when price is high, taxis will avoid joining the queue when there's some other cars waiting, thus cause a congestion of passengers.

B. Pricing Strategy

We measured the performance of the pricing strategy implied by Proposition 5, i.e. to charge a taxi driver with price $p_{M,N}$ upon entering the system when $p_{M,N} > 0$, and set p to be high when $p_{M,N} \leq 0$. Fig. 3 illustrates the accumulate profit when $\lambda_2 = 0.5$ and $\lambda_2 = 2$ with different pricing strategy.

We can tell from Fig. 3 that our state-dependent pricing strategy gives the best performance when the arrival rate of passengers is large enough, but it's suboptimal when the arrival rate of passenger is small. One possible explanation is that in the beginning, the profit rate obtained by state-dependent pricing strategy when $\lambda_2 = 0.5$ is no worse than fixed pricing strategy, which is illustrated in Fig. 3, yet as time goes on, the state-dependent pricing strategy will attract as many taxis as possible to join a queue with very low price, thus cause a



(a) $\lambda_2 = 2$ (b) $\lambda_2 = 0.5$

Fig. 3. Earning rate of the system with different pricing strategy. We can tell that our state-dependent pricing strategy gives the best performance when the arrival rate of passengers is large enough, but it's suboptimal when the arrival rate of passenger is small.

congestion with the number of taxis outweighing the number of passengers, which turns other taxis away and causes the firm to profit less than fixed pricing strategy. This confirms the necessity of the two conditions required by Proposition 5. Therefore, it is recommended that the firm adopts the state-dependent pricing strategy when the arrival rate of passengers is high, and adopts a fixed pricing strategy when the arrival rate of passengers is low.

VI. Conclusion

In this work, we investigated the taxi-passenger matching process in a double-ended queueing system, with non-zero matching time. We proposed a method for computing the Laplacian transform of conditional response time when the distribution of matching time and inter-arrival time between passengers is general, and characterize the value function of the pricing strategy for the firm when the system is Markovian. Experiment results show the correctness and effectiveness of our proposed method. However, many topics are still left unexplored, such as the close form for the expression of mean response time in a double-ended queueing system, the effect of batch size in the system and the decision modeling on the passenger side. We wish future work would shed more light to the truth of double-ended queueing system.

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