RETHINKING STRATEGIC TRANSITIVITY IN TWO-PLAYER SYMMETRIC ZERO-SUM GAME

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ABSTRACT

In this work, I define a class of two-player symmetric non-zero game with a certain transitive properties between strategies associated with two functions f and g, which I call (f, g)-transitive games. Conditioned on several specifications of f and g, I studied some properties of such kind of transitive games and discussed their relationship with potential games. A hierarchy of transitive games and potential games are finally constructed after ample study.

1 Introduction

Transitivity, in some way, is a fundamental concept in game theory. One basic assumption for player's preference of outcome is that the preference order is transitive, i.e., if a player prefers outcome o_1 to o_2 and prefers outcome o_2 to o_3, then she would certainly prefer o_1 to o_3.

However, such transitivity assumption usually doesn't hold for a player's strategies. A typical example is the game of Rock-Paper-Scissors, Rock can beat Scissors, Scissors can beat Paper, but it doesn't imply Rock can beat Paper. Such kind of cyclic conquering is also a common situation in many real-life competitive games. For example, in football games, which can be viewed as an extensive form game, team A may win team B, team B may win team C, but team C with their specific strategy can also have a chance to win team A. Also, for the game of Texas holdem poker, which can be viewed as a Bayesian game, one agent with its aggressive strategy and rough estimation of its own hand strength, may beat another agent who just randomly plays some actions, and one agent who carefully estimated its opponent's card strength and its own winning probability may win the bold agent. But the careful agent can still lose money to a random agent.

Such games where the dominance between strategies are not transitive are typically named nontransitive games. Several previous work has been dedicated to study nontransitive games [8, 7, 5], however, up to my best knowledge, there has been basically no work that formally studies the class of games with certain strategic transitivity property. It might be because researchers naturally assume transitivity is a common property not worth studying, yet as the situations in football game and Texas holdem poker imply, nontransitive is the real common situation. To make matter's worse, nontransitive may cause extreme unfairness, especially when we need to give a total order for agents' strategies but the comparison between strategies isn't even transitive (**e.g. try to score students' poker project based on mutual competition**). Therefore, it's necessary to study games with strategic transitive property, which may guide real-life game designers to design more fair mechanisms.

In this work, I define a class of two-player symmetric non-zero game with a certain transitive properties associated with two functions f and g, which I call (f, g)-transitive games. With several specifications of f and g, I studied some properties of such kind of transitive games and discussed their relationship with potential games [6]. A hierarchy of transitive games and potential games are finally constructed after ample study.

2 Related work

2.1 Two-player symmetric zero-sum game

Two-player symmetric zero-sum game is a class of games that enjoys some good symmetry properties and could be used to model many games such as football games or Rock-Paper-Scissors. Work [4] studied the existence of pure-strategy Nash equilibrium in two-player symmetric zero-sum game, and discovered that the existence of pure-strategy Nash equilibrium is equivalent to that the game is not generalized Rock-Paper-Sciessors. Work [2] analyzed the distribution of optimal strategies in two-player symmetric zero-sum game and determined the probability for an optimality of a randomized strategy when the game is drawn from a distribution that satisfies certain regularity conditions.

2.2 Nontransitive game

Nontransitive game is a class of game where strategies may cyclically dominate between each other. Work [7, 5, 8] studied a special nontransitive game called nontransitive dice. Work [1] proved that a two-player symmetric zero-sum game can be decomposed into two parts, transitive part and cyclic part, and proposed learning algorithms to learn best responses on these two components respectively.

2.3 Potential game

Potential game is a class of games whose utility function admits a potential function. It was first proposed in [6], and work [9] gives out a method to decide whether a game is an oridnal potential game.

3 Preliminaries

In this section, I will first give a formal definition for two-player symmetric zero-sum game, which is the basic setting of this work, and then try to provide a unified definition for classes of symmetric zero-sum game that possesses a certain transitivity property in strategies.

3.1 Basic definitions and previous results

A two-player game can be characterized as a tuple $G = (A_1, A_2, u_1, u_2)$, where A_1 is the set of pure actions (strategies) for the first player and A_2 is the set of pure actions (strategies) for the second player. u_1 and u_2 are both maps from $A_1 \times A_2$ to a real number, indicating the utilities for the first player and the second player respectively. In this work, it's not necessary for A_1 or A_2 to be finite.

A two-player symmetric zero-sum game is a two-player game with some special properties, which is formally defined as follows:

Definition 1 (Two-player symmetric zero-sum game)**.** *A two-player game $G = (A_1, A_2, u_1, u_2)$ is said to be a two-player symmetric zero-sum game if it satisfies:*

1. $A_1 = A_2$
2. $u_1 + u_2 = 0$
3. $\forall a_1 \in A_1, a_2 \in A_2$, $u_1(a_1, a_2) = u_2(a_2, a_1)$

In the definition, condition *1.* says two players involved in the game can take exactly the same set of actions or strategies, condition *2.* says the game is zero-sum and condition *3.* says no matter the player is in the first position or the second, when the action profiles are the same, she would receive the same amount of payoff. From now on, I will use (A, u) to denote a two-player symmetric zero-sum game, where A is the action set for both players and u is the utility function for the first player.

Several observations can be induced from the definition of two-player symmetric zero-sum game:

Observation 1. *If $G = (A, u)$ is a two-player symmetric zero-sum game, then $\forall a_1, a_2 \in A$, $u(a_1, a_2) + u(a_2, a_1) = 0$.*

Observation 2. *If $G = (A, u)$ is a two-player symmetric zero-sum game, then $\forall a \in A$, $u(a, a) = 0$.*

Strong and strict as the definition looks, two-player symmetric zero-sum games can actually model many real life games such as chess, GO and football game where the two players are equal and if one player wins, the other player will lose.

the game of Rock-Paper-Scissors (RPS) can also be characterized as a two-player symmetric zero-sum game, and RPS can be generalized into the following game named gRPS:

Definition 2 (generalized Rock-Paper-Scissors, gRPS)**.** *A two-player symmetric zero-sum game (A, u) is a generalized Rock-Paper-Scissors (gRPS) if*

$$\forall a_2 \in A, \exists a_1 \in A, u(a_1, a_2) > 0$$

It has been proved in [4] that gRPS can partition two-player symmetric zero-sum game into two classes, one that possesses a pure-strategy Nash equilibrium, the other that doesn't:

Theorem 1 ([4])**.** *A two-player symmetric zero-sum game (A, u) possesses a pure-strategy Nash equilibrium if and only if it is not a gRPS.*

Proof. If (A, u) has no pure-strategy Nash equilibrium, then $\forall a_2 \in A$, (a_2, a_2) is not a pure-strategy Nash equilibrium, so we can find a deviation (a_1, a_2) such that $u(a_1, a_2) > u(a_2, a_2) = 0$, so (A, u) is a gRPS. Conversely, if (A, u) possesses a pure-strategy Nash equilibrium, which is denoted by (a_1, a_2), then we must have $u(a_1, a_2) = 0$, since if $u(a_1, a_2) \neq 0$, without a loss of generality suppose $u(a_1, a_2) > 0$, then the second player will deviate to (a_1, a_1) to better respond to the first player. And since (a_1, a_2) is a pure-strategy Nash equilibrium, we can not find $a_3 \in A$ such that $u(a_3, a_2) > 0 = u(a_1, a_2)$, so (A, u) is not a gRPS. $\square$

Theorem 1 provides a technical tool to analyze whether a specified two-player symmetric zero-sum game has a pure-strategy Nash equilibrium.

3.2 Unified definition for transitive games

I define (f, g)-transitive game in two-player symmetric zero-sum game as follows:

Definition 3 ((f, g)-transitive game)**.** *Two-player symmetric zero-sum game $(A.u)$ is a (f, g)-transitive game if:*

$$\forall\ a_1, a_2, a_3 \in A,\ f(a_1, a_2) = g((a_1, a_3), (a_3, a_2))$$

where $f : A \times A \to \mathbb{R}$ and $g : (A \times A) \times (A \times A) \to \mathbb{R}$

The intuition behind this definition is that if a game (A, u) enjoys some transitive properties between strategies, then the relationship between strategy profiles (a_1, a_3) and (a_3, a_2), which is characterized by g, can reveal some properties of strategy profile (a_1, a_2), which is characterized by f.

Notice that when $f = g = c$, where c is some constant real number, then (f, g)-transitive game is just two-player symmetric zero-sum game. However, when f and g is non-degenerate, (f, g)-transitive game can be non-trivial, which we will see in later sections.

4 Exact transitive game

In this section, I propose to consider a special class of two-player symmetric zero-sum game, called exact transitive game, whose utility function enjoys a nice transitive property. I would first introduce the definition and some basic properties about exact transitive game, and then discuss the relationship between exact transitive game and exact potential game.

4.1 Basic definition and properties

Exact transitive game is formally defined as follows:

Definition 4 (Exact transitive game)**.** *A two-player symmetric zero-sum game (A, u) is an exact transitive game, if*

$$\forall a_1, a_2, a_3 \in A, u(a_1, a_3) + u(a_3, a_2) = u(a_1, a_2)$$

Notice that exact transitive game is (f, g)-transitive game where $f(a_1, a_2) = u(a_1, a_2)$ and $g((a_1, a_2), (a_3, a_4)) = u(a_1, a_2) + u(a_3, a_4)$. Intuitively, the utility function u in an exact transitive game measures some kind of difference between two strategies. Taking it as a analogue to the 'minus' operation between real numbers, for any $a, b, c \in \mathbb{R}$, it is natural to have $(a - c) + (c - b) = a - b$. The following proposition justifies such intuition:

Proposition 1. *Game (A, u) is an exact transitive game if and only if there exists a map $f : A \to \mathbb{R}$, such that:*

$$\forall a_1, a_2 \in A, u(a_1, a_2) = f(a_1) - f(a_2)$$

Proof. $\Leftarrow$: If (A, u) satisfies $u(a_1, a_2) = f(a_1) - f(a_2)$ for any two actions $a_1, a_2 \in A$, then we have:

$$\begin{aligned} u(a_1, a_3) + u(a_3, a_2) &= f(a_1) - f(a_3) + f(a_3) - f(a_2) \\ &= f(a_1) - f(a_2) \\ &= u(a_1, a_2) \end{aligned}$$

therefore (A, u) is an exact transitive game.

$\Rightarrow$: If (A, u) is an exact transitive game, we take a fixed action $a^* \in A$, and let $f(a) = u(a, a^*)$, then for any $a_1, a_2 \in A$, we have:

$$u(a_1, a_2) = u(a_1, a^*) + u(a^*, a_2) = u(a_1, a^*) - u(a_2, a^*) = f(a_1) - f(a_2)$$

□

With proposition 1, we can see my definition about exact transitive game is consistent with the transitive component defined in [1], therefore the theorem of game decomposition can be applied here:

Theorem 2 ([1])**.** *For every two-player symmetric zero-sum game (A, u), it can be decomposed into two symmetric games $(A, u) = (A_1 \cap A_2, u_1 + u_2)$, such that (A_1, u_1) is an exact transitive game and (A_2, u_2) is a game satisfying:*

$$\forall x \in A_2, \int_{A_2} u_2(x, y) \mathrm{d}y = 0$$

One problem worth considering is whether there's a pure-strategy Nash equilibrium in an exact transitive game, and the following proposition provides an answer:

Theorem 3. *Suppose (A, u) is an exact transitive game and $u(a_1, a_2) = f(a_1) - f(a_2)$ for any $a_1, a_2 \in A$. Then (A, u) possesses a pure-strategy Nash equilibrium if and only if*

$$\exists\, a^* \in A,\ f(a^*) = \sup_{a \in A} f(a)$$

Proof. $\Rightarrow$: If (A, u) has a pure-strategy Nash equilibrium, denoted by (a_1, a_2). By the argument in theorem 1, we know $u(a_1, a_2) = 0$, implying $f(a_1) = f(a_2)$. We let $a^* = a_2$, since there's no profitable deviation for the first player, we have for all $a \in A$, $f(a) \leq f(a^*)$, therefore $f(a^*) \geq \sup_{a \in A} f(a)$. Also we have $f(a^*) \leq \sup_{a \in A} f(a)$ since $a^* \in A$, so $f(a^*) = \sup_{a \in A} f(a)$.

$\Leftarrow$: Suppose we can find $a^* \in A$ such that $f(a^*) = \sup_{a \in A} f(a)$, then by definition $\forall a \in A, f(a) \leq f(a^*)$, so for any $a \in A$:

$$u(a, a^*) = f(a) - f(a^*) \leq f(a^*) - f(a^*) = 0$$

Therefore, (A, u) is not a gRPS. Then by theorem 1, we know (A, u) has a pure-strategy Nash equilibrium. □

From the results in theorem 3, we can easily deduce that every finite exact transitive game possesses a pure-strategy Nash equilibrium.

4.2 Exact transitive game and exact potential game

Monderer and Shapley introduces a class of games with potential function defined on strategy profiles in [6], including a class of game named exact potential game. Here I give the definition of exact potential game for two-player games:

Definition 5 (Exact potential game [6])**.** *Two-player game (A_1, A_2, u_1, u_2) is an exact potential game if there exists a potential function $P : A_1 \times A_2 \to \mathbb{R}$, such that:*

$$\forall a_1, \tilde{a}_1 \in A_1, a_2 \in A_2,\ u_1(a_1, a_2) - u_1(\tilde{a}_1, a_2) = P(a_1, a_2) - P(\tilde{a}_1, a_2)$$
$$\forall a_1 \in A_1, a_2, \tilde{a}_2 \in A_2,\ u_2(a_1, a_2) - u_2(a_1, \tilde{a}_2) = P(a_1, a_2) - P(a_1, \tilde{a}_2)$$

With the definition of two-player symmetric zero-sum game, the following observation is obvious:

Observation 3. *Two-player symmetric zero-sum game (A, u) is an exact potential game if and only if $\forall a_1, a_2, a_3 \in A$:*

$$u(a_1, a_3) - u(a_2, a_3) = P(a_1, a_3) - P(a_2, a_3)$$
$$u(a_1, a_3) - u(a_2, a_3) = P(a_3, a_1) - P(a_3, a_2)$$

By proposition 1 we know the utility function $u(a_1, a_2)$ for an exact transitive game (A, u) can be separated as a difference $f(a_1) - f(a_2)$. In this sense, f can be viewed as a potential function. Notice that in exact potential game, potential function P is defined on strategy profiles yet in exact transitive game, f is defined on strategies, which at first sight, seems to be different. However the following theorem shows that exact potential game and exact transitive game are actually equivalent on two-player symmetric zero-sum game.

Theorem 4. *Two-player symmetric zero-sum game (A, u) is an exact transitive game if and only if it is an exact potential game.*

Proof. $\Rightarrow$: Suppose (A, u) is an exact transitive game, then by proposition 1 we can find a function $f : A \to \mathbb{R}$ such that $u(a_1, a_2) = f(a_1) - f(a_2)$ for every $a_1, a_2 \in A$. We define $P(a_1, a_2) = f(a_1) + f(a_2)$, then $\forall a_1, a_2, a_3 \in A$:

$$\begin{aligned}
P(a_1, a_3) - P(a_2, a_3) &= f(a_1) + f(a_3) - f(a_2) - f(a_3) \\
&= f(a_1) - f(a_3) + f(a_3) - f(a_2) \\
&= u(a_1, a_3) + u(a_3, a_2) \\
&= u(a_1, a_3) - u(a_2, a_3) \\
P(a_3, a_1) - P(a_3, a_2) &= f(a_3) + f(a_1) - f(a_3) - f(a_2) \\
&= f(a_1) - f(a_3) + f(a_3) - f(a_2) \\
&= u(a_1, a_3) + u(a_3, a_2) \\
&= u(a_1, a_3) - u(a_2, a_3)
\end{aligned}$$

which is consistent with observation 3.[1]

$\Leftarrow$: Suppose (A, u) is not an exact transitive game, then by definition we can find $a_1, a_2, a_3 \in A$, such that:

$$u(a_1, a_3) + u(a_3, a_2) \neq u(a_1, a_2) \tag{1}$$

Assume (A, u) is an exact potential game, and let P be its potential function, then we have:

$$\begin{aligned}
u(a_1, a_2) - u(a_1, a_3) &= P(a_1, a_3) - P(a_1, a_2) \\
u(a_2, a_3) - u(a_1, a_3) &= P(a_2, a_3) - P(a_1, a_3) \\
u(a_2, a_3) - u(a_2, a_2) &= P(a_2, a_2) - P(a_2, a_3) \\
u(a_1, a_2) - u(a_2, a_2) &= P(a_1, a_2) - P(a_2, a_2)
\end{aligned}$$

Summing them all up we have:

$$2\left(u(a_1, a_2) - u(a_1, a_3) + u(a_2, a_3)\right) = P(a_1, a_2) - P(a_1, a_2) = 0$$

which implies:

$$u(a_1, a_2) = u(a_1, a_3) + u(a_3, a_2) \tag{2}$$

(1) and (2) lead to a contradiction. Therefore (A, u) is not an exact potential game. □

5 Ordinal transitive game

In this section, I try to generalize exact transitive game into another class of (f, g)-transitive game, called ordinal transitive game. I will first introduce the definition and some basic properties of ordinal potential game, then I will discuss the relationship between ordinal transitive game and ordinal potential game, finally I will introduce a subclass of ordinal transitive game named regulated ordinal transitive game, which enjoys some nice properties.

5.1 Basic definition and properties

An ordinal transitive game is a two-player symmetric zero-sum game formally defined as follows:

Definition 6 (Ordinal transitive game)**.** *Two-player symmetric zero-sum game (A, u) is an ordinal transitive game if:*

$$\forall\, a_1, a_2, a_3 \in A,\ u(a_1, a_2) > 0 \Leftrightarrow u(a_1, a_3) + u(a_3, a_2) > 0$$

Notice that ordinal transitive game is a (f, g)-transitive game where $f(a_1, a_2) = [u(a_1, a_2) > 0]$ and $g((a_1, a_2), (a_3, a_4)) = [u(a_1, a_2) + u(a_3, a_4) > 0]$. From the definition of ordinal transitive game, we have the following observation:

[1] This part of proof can be viewed as a corollary of theorem 1 in [3]

Observation 4. *Two-player symmetric zero-sum game* (A, u) *is an ordinal transitive game if and only if:*

$$\forall\, a_1, a_2, a_3 \in A,\ u(a_1, a_2) > 0 \Rightarrow u(a_1, a_3) + u(a_3, a_2) > 0$$

This is because $\forall\, a_3 \in A, u(a_1, a_3) + u(a_3, a_2) > 0$ tautologically implies $u(a_1, a_2) > 0$ by setting $a_3 = a_2$. The following proposition further reveals a certain insight indicated by the definition.

Proposition 2. *If* (A, u) *is an ordinal transitive game, define a relation* $\approx$ *on* A *as:*

$$\forall\, a_1, a_2 \in A,\ a_1 \approx a_2 \Leftrightarrow u(a_1, a_2) = 0$$

then $\approx$ *is an equivalence relation.*

Proof. for refelxivity, by definition of two-player symmetric zero-sum game, we know $\forall a \in A$, $u(a, a) = 0$, so $a \approx a$.

for symmetry, by definition of two-player symmetric zero-sum game, we know $\forall a_1, a_2 \in A$, $u(a_1, a_2) = -u(a_2, a_1)$, so if $u(a_1, a_2) = 0$, we must have $u(a_2, a_1) = 0$, therefore $a_1 \approx a_2$ implies $a_2 \approx a_1$.

for transitivity, suppose we have $u(a_1, a_2) = 0$ and $u(a_2, a_3) = 0$. Suppose $u(a_1, a_3) \neq 0$, without a loss of generality, assume $u(a_1, a_3) > 0$, by definition of ordinal transitive game, we have $u(a_1, a_2) + u(a_2, a_3) > 0$, a contradiction. Therefore we must have $u(a_1, a_3) = 0$. ☐

With proposition 2 we are able to define equivalence class on the action set for an ordinal transitive game.

Definition 7. *For an ordinal transitive game* (A, u)*, for every* $a \in A$*, define* $\overline{a} = \{a' \in A \mid u(a', a) = 0\}$

Notice that $\{\overline{a} \mid a \in A\}$ is a partition of A. Now I try to define a binary relation on $\{\overline{a} \mid a \in A\}$.

Definition 8. *For an ordinal transitive game* (A, u)*, define* $\geq$ *on* $\{\overline{a} \mid a \in A\}$ *as:*

$$\forall\, a_1, a_2 \in A,\ \overline{a_1} \geq \overline{a_2} \Leftrightarrow u(a_1, a_2) \geq 0$$

The following proposition justifies definition 8.

Proposition 3. $\geq$ *defined in definition 8 is a linear order on* $\{\overline{a} \mid a \in A\}$

Proof. We first show that $\geq$ is well defined. For any $a_1, \tilde{a}_1 \in \overline{a_1}$ and $a_2, \tilde{a}_2 \in \overline{a_2}$, if $u(a_1, a_2) = 0$, then $\overline{a_1} = \overline{a_2}$, so we have $u(\tilde{a}_1, \tilde{a}_2) = 0$. If $u(a_1, a_2) > 0$, then by definition we know $0 = u(a_1, \tilde{a}_1) > u(a_2, \tilde{a}_1)$, therefore $u(\tilde{a}_1, a_2) > 0$, so $u(\tilde{a}_1, \tilde{a}_2) > u(a_2, \tilde{a}_2) > 0$. Therefore $u(a_1, a_2) \geq 0 \Leftrightarrow u(\tilde{a}_1, \tilde{a}_2) \geq 0$.

Then, for anti-symmetry, we have if $\overline{a_1} \geq \overline{a_2}$ and $\overline{a_2} \geq \overline{a_1}$, then $u(a_1, a_2) \geq 0$ and $u(a_1, a_2) \leq 0$, so $u(a_1, a_2) = 0$, so $\overline{a_1} = \overline{a_2}$. For reflexivity, since $u(a_1, a_2) = 0$ if $\overline{a_1} = \overline{a_2}$, we must have $\overline{a_1} \geq \overline{a_2}$. For transitivity, if $\overline{a_1} \geq \overline{a_2}$ and $\overline{a_2} \geq \overline{a_3}$, then $u(a_1, a_2) \geq 0$ and $u(a_2, a_3) \geq 0$. If $u(a_1, a_2) > 0$, we have $u(a_1, a_3) > u(a_2, a_3) \geq 0$, so $\overline{a_1} \geq \overline{a_3}$; if $u(a_1, a_2) = 0$, then $\overline{a_1} = \overline{a_2}$, so $\overline{a_1} \geq \overline{a_3}$. ☐

With proposition 3, I am able to answer the question about the existence of a pure-strategy Nash equilibrium in an ordinal transitive game possesses.

Theorem 5. *An ordinal transitive game* (A, u) *possesses a pure-strategy Nash equilibrium if and only if there exists a maximal element in* $\{\overline{a} \mid a \in A\}$ *under the* $\geq$ *relation defined in 8, i.e.* $\exists\, \overline{a^*} \in \{\overline{a} \mid a \in A\}$*, such that* $\forall\, a \in A$, $\overline{a^*} \geq \overline{a}$.

Proof. $\Rightarrow$: If (A, u) has a pure-strategy Nash equilibrium, denoted by (a_1, a_2). By the argument in theorem 1, we know $u(a_1, a_2) = 0$, therefore $\overline{a_1} = \overline{a_2}$. We let $a^* = a_1$, since there's no profitable deviation for the first player, we have for all $a \in A$, $u(a^*, a) \geq 0$, so $\overline{a^*} \geq \overline{a}$, therefore $\overline{a^*}$ is the maximal element.

$\Leftarrow$: Suppose we can find $a^* \in A$ such that $\overline{a^*} \geq \overline{a}$ for any $a \in A$, then by definition $\forall a \in A, u(a^*, a) \geq 0$, equivalent to $u(a, a^*) \leq 0$. Therefore, (A, u) is not a gRPS. Then by theorem 1, we know (A, u) has a pure-strategy Nash equilibrium. ☐

By the results in theorem 5, we can deduce that if an ordinal transitive game (A, u) is finite, then (A, u) possesses a pure-strategy Nash equilibrium.

5.2 Ordinal transitive game and ordinal potential game

There's another class of potential game other than exact potential game, named ordinal potential game. On two-player games, ordinal potential game is formally defined as follows:

Definition 9 (Ordinal potential game [6]). *A two-player game (A_1, A_2, u_1, u_2) is an ordinal potential game if there exists $P : A_1 \times A_2 \to \mathbb{R}$, such that*

$$\forall a_1, \tilde{a}_1 \in A_1, a_2 \in A_2,\ u_1(a_1, a_2) - u_1(\tilde{a}_1, a_2) > 0 \Leftrightarrow P(a_1, a_2) - P(\tilde{a}_1, a_2) > 0$$

$$\forall a_1 \in A_1, a_2, \tilde{a}_2 \in A_2,\ u_2(a_1, a_2) - u_2(a_1, \tilde{a}_2) > 0 \Leftrightarrow P(a_1, a_2) - P(a_1, \tilde{a}_2) > 0$$

On two-player symmetric zero-sum game, I have the following observation:

Observation 5. *Two-player symmetric zero-sum game (A, u) is an ordinal potential game if and only if $\forall a_1, a_2, a_3 \in A$:*

$$u(a_1, a_3) - u(a_2, a_3) > 0 \Leftrightarrow P(a_1, a_3) - P(a_2, a_3) > 0$$

$$u(a_1, a_3) - u(a_2, a_3) > 0 \Leftrightarrow P(a_3, a_1) - P(a_3, a_2) > 0$$

Based on the equivalence result in theorem 4, one may naturally assume that ordinal transitive game and ordinal potential game are equivalent. However, the following proposition shows that these two classes of games are not equivalent, and there's even no containing relationship between them.

Proposition 4. *In two-player symmetric zero-sum games, an ordinal transitive game is not necessarily an ordinal potential game, and an ordinal potential game is not necessarily an ordinal transitive game.*

Proof. Consider a two-player symmetric zero-sum game (A, u) such that $A = \{a_1, a_2, a_3, a_4\}$ and u is defined in the following matrix:

$$\begin{array}{c c} & \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

the first player chooses a row and the second player chooses a column.

Through case-by-case checking we can see (A, u) is an ordinal transitive game, however I claim it's not an ordinal potential game. Assume (A, u) is a ordinal potential game let P be the potential function, since $u(a_2, a_3) > u(a_1, a_3)$, we have $P(a_2, a_3) > P(a_1, a_3)$. Since $u(a_4, a_2) > u(a_3, a_2)$, so we have $P(a_2, a_4) > P(a_2, a_3)$. Since $u(a_1, a_4) \geq u(a_2, a_4)$, we have $P(a_1, a_4) \geq P(a_2, a_4)$. Since $u(a_3, a_1) \geq u(a_4, a_1)$, we have $P(a_1, a_3) \geq P(a_1, a_4)$, so we finally get $P(a_1, a_3) > P(a_1, a_3)$, a contradiction.

Consider another two-player symmetric zero-sum game (A, u) such that $A = \{a_1, a_2, a_3\}$ and u is defined in the following matrix:

$$\begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix} \end{array}$$

the first player chooses a row and the second player chooses a column. We define a potential function P as the following matrix:

$$\begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 0 & -1 & -5 \\ -1 & -2 & -3 \\ -5 & -3 & -4 \end{pmatrix} \end{array}$$

Through case-by-case checking we know P is a valid potential function and (A, u) is an ordinal potential game, but it's not a ordinal transitive game since $u(a_1, a_2) > 0$, but $u(a_1, a_3) + u(a_3, a_2) = -1 < 0$. □

5.3 Regulated ordinal transitive game

Now I define a subclass of ordinal transitive game called regulated ordinal transitive game:

Definition 10 (Regulated ordinal transitive game)**.** *Two-player symmetric zero-sum game (A, u) is a regulated ordinal transitive game if:*

$$\forall\, a_1, a_2, a_3 \in A,\ u(a_1, a_2) > 0 \Leftrightarrow u(a_1, a_3) + u(a_3, a_2) > 0$$
$$\forall\, a_1, a_2, a_3 \in A,\ u(a_1, a_2) \geq 0 \Leftrightarrow u(a_1, a_3) + u(a_3, a_2) \geq 0$$

Notice that a regulated ordinal transitive game is a (f, g)-transitive game where $f(a_1, a_2) = 2[u(a_1, a_2) > 0] + [u(a_1, a_2) \geq 0]$ and $g((a_1, a_2), (a_3, a_4)) = 2[u(a_1, a_2) + u(a_3, a_4) > 0] + [u(a_1, a_2) + u(a_3, a_4) \geq 0]$. One can also observe from the definition that:

Observation 6. *If (A, u) is a regulated ordinal transitive game, then $\forall\, a_1, a_2 \in A$*

$$u(a_1, a_2) = 0 \Leftrightarrow (\exists a_3 \in A)(u(a_1, a_3) + u(a_3, a_2) = 0)$$

Given an action set A, I now denote $\{\overline{a} \mid \forall\, a \in A\}$ as $\overline{A}$. Regulated ordinal transitive game enjoys a property that one can naturally define a partial order on $\overline{A} \times \overline{A}$ using u.

Definition 11. *For a regulated ordinal transitive game (A, u), define a binary relation $\geq$ on $\overline{A} \times \overline{A}$ as for any $(\overline{a_1}, \overline{a_2}), (\overline{a_3}, \overline{a_4}) \in \overline{A} \times \overline{A}$, $(\overline{a_1}, \overline{a_2}) \geq (\overline{a_3}, \overline{a_4})$ if $u(a_1, a_2) \geq u(a_3, a_4)$ and $a_2 = a_4$ or $u(a_2, a_1) \geq u(a_4, a_3)$ and $a_1 = a_3$.*

The following lemma justifies such definition:

Lemma 1. *$\geq$ defined in definition 11 is a partial order on $\overline{A} \times \overline{A}$.*

Proof. First notice that for any $a_1, a_2 \in A$, and any $\tilde{a}_1 \in \overline{a_1}$ and $\tilde{a}_2 \in \overline{a_2}$, we have $u(a_1, a_2) = u(\tilde{a}_1, \tilde{a}_2)$. This is because by observation 6, $u(a_1, a_2) = u(\tilde{a}_1, a_2)$ and $u(\tilde{a}_2, \tilde{a}_1) = u(a_2, \tilde{a}_1)$. Therefore $\geq$ is well defined.

For anti-symmetry, we can see if $(\overline{a_1}, \overline{a_2}) \geq (\overline{a_3}, \overline{a_4})$ and $(\overline{a_3}, \overline{a_4}) \geq (\overline{a_1}, \overline{a_2})$, assume $a_2 = a_4$ we have $u(a_1, a_2) \geq u(a_3, a_4)$ and $u(a_1, a_3) \leq u(a_3, a_4)$, so $u(a_1, a_2) = u(a_3, a_4)$, and by observation 6 we know $\overline{a_1} = \overline{a_3}$. For reflexivity, we know if $\overline{a_1} = \overline{a_3}$ and $\overline{a_2} = \overline{a_4}$, we have $u(a_1, a_2) = u(a_3, a_4) \geq u(a_3, a_4)$, so we have $(\overline{a_1}, \overline{a_2}) \geq (\overline{a_3}, \overline{a_4})$. And we can directly use the transitivity of $\geq$ defined on real number to prove the transitivity of $\geq$ defined on $\overline{A} \times \overline{A}$. □

Then I'm about to discuss the relationship between regulated ordinal transitive game and ordinal potential game. I first introduce the concept about weak improvement cycle.

Definition 12 (Weak improvement cycle [9])**.** *For a two-player symmetric zero-sum game (A, u), a weak improvement cycle is a sequence $\{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$, such that $(a_i, b_i) \in A \times A$ for all $1 \leq i \leq m$ and $(a_m, b_m) = (a_1, b_1)$, and for all $1 \leq i < m$, either $a_i = a_{i+1}$ or $b_i = b_{i+1}$. Furthermore, for $1 \leq i < m$, if $a_i = a_{i+1}$, $u(a_{i+1}, b_{i+1}) \leq u(a_i, b_i)$ and if $b_i = b_{i+1}$, $u(a_{i+1}, b_{i+1}) \geq u(a_i, b_i)$, and there exists at least one integer i for the inequality to strictly hold.*

The following lemma shows the necessary and sufficient condition for a two-player symmetric zero-sum game to be an ordinal potential game when the action set is countable.

Lemma 2. *A two-player symmetric zero-sum game (A, u) with a countable A is an ordinal potential game if and only if there's no strict improvement cycle for (A, u).*

With lemma 1 and lemma 2, the following theorem reveals the relationship between regulated ordinal transitive game and ordinal potential game.

Theorem 6. *When the action set is countable, every regulated ordinal transitive game (A, u) is an ordinal potential game.*

Proof. Suppose there's a regulated ordinal transitive game (A, u) with countable A that is not an ordinal potential game, then by lemma 2 there's a weak improvement cycle $\{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m) = (a_1, a_2)\}$. Without a loss of generality, suppose $b_1 = b_2$ and $u(a_1, b_1) < u(a_2, b_2)$. Now let $>$ denote the strict partial order induced by $\geq$ defined on $\overline{A} \times \overline{A}$, then we have $(\overline{a_2}, \overline{b_2}) > (\overline{a_1}, \overline{b_1})$. By definition of weak improve cycle, we have

$$(\overline{a_1}, \overline{b_1}) = (\overline{a_m}, \overline{b_m}) \geq (\overline{a_{m-1}}, \overline{b_{m-1}}) \geq \cdots \geq (\overline{a_2}, \overline{b_2})$$

By transitivity proved in lemma 1, we have $(\overline{a_1}, \overline{b_1}) > (\overline{a_1}, \overline{b_1})$, which contradicts to the irreflexivity of strict partial order. □

6 Weak transitive game

In this section, I further generalize ordinal transitive game into a class of two-player symmetric zero-sum game, called weak transitive game. I will first introduce the definition and some properties of weak transitive game, and discuss it's relationship with ordinal potential game, which leads to a hierarchy of potential games and transitive games.

6.1 Basic definitions and properties

A weak transitive game is formally defined as follows:

Definition 13 (Weak transitive game)**.** *Two-player symmetric zero-sum game (A, u) is a weak transitive game if:*

$$\forall\, a_1, a_2, a_3 \in A,\ u(a_1, a_2) \geq 0 \Leftrightarrow (u(a_2, a_3) \geq 0 \rightarrow u(a_1, a_3) \geq 0)$$

Notice that weak transitive game is also a (f, g)-transitive game, where $f(a_1, a_2) = [u(a_1, a_2) \geq 0]$ and $g((a_1, a_2), (a_3, a_4)) = 1 - [u(a_4, a_3) \geq 0](1 - [u(a_1, a_2) \geq 0])$. Also it's easy to see ordinal transitive game is a subclass of weak transitive game. However, even though the definition for ordinal transitive game is stronger, some properties of ordinal transitive game still holds for weak transitive game.

Proposition 5. *If (A, u) is a weak transitive game, define a relation $\approx$ on A as:*

$$\forall\, a_1, a_2 \in A,\ a_1 \approx a_2 \Leftrightarrow u(a_1, a_2) = 0$$

then $\approx$ is an equivalence relation.

Proof. for refelxivity, by definition of two-player symmetric zero-sum game, we know $\forall a \in A$, $u(a, a) = 0$, so $a \approx a$.

for symmetry, by definition of two-player symmetric zero-sum game, we know $\forall a_1, a_2 \in A$, $u(a_1, a_2) = -u(a_2, a_1)$, so if $u(a_1, a_2) = 0$, we must have $u(a_2, a_1) = 0$, therefore $a_1 \approx a_2$ implies $a_2 \approx a_1$.

for transitivity, suppose we have $u(a_1, a_2) = 0$ and $u(a_2, a_3) = 0$, so we have $u(a_1, a_2) \geq 0$ and $u(a_2, a_3) \geq 0$, therefore, by definition, we have $u(a_1, a_3) \geq 0$, also since $u(a_3, a_2) \geq 0$ and $u(a_2, a_1) \geq 0$, we have $u(a_3, a_1) \geq 0$, therefore $u(a_1, a_3) = 0$, so $a_1 \approx a_3$. ☐

With proposition 5 we are able to define equivalence class on the action set for a weak transitive game. For any $a \in A$, I define $\overline{a}$, $\overline{A}$ and $\geq$ on $\overline{A}$ the same way as I define them for ordinal transitive game.

Proposition 6. *$\geq$ defined in definition 8 is still a linear order even the game (A, u) is a weak transitive game.*

Proof. We first show that $\geq$ is well defined. For any $a_1, \tilde{a}_1 \in \overline{a_1}$ and $a_2, \tilde{a}_2 \in \overline{a_2}$, if $u(a_1, a_2) = 0$, then $\overline{a_1} = \overline{a_2}$, so we have $u(\tilde{a}_1, \tilde{a}_2) = 0$. If $u(a_1, a_2) > 0$, since $u(\tilde{a}_1, a_1) = 0 \geq 0$, we have $u(\tilde{a}_1, a_2) \geq 0$, also since $u(a_2, \tilde{a}_2) = 0 \geq 0$, we have $u(\tilde{a}_1, \tilde{a}_2) \geq 0$. Therefore $u(a_1, a_2) \geq 0 \Leftrightarrow u(\tilde{a}_1, \tilde{a}_2) \geq 0$.

Then, for anti-symmetry, we have if $\overline{a_1} \geq \overline{a_2}$ and $\overline{a_2} \geq \overline{a_1}$, then $u(a_1, a_2) \geq 0$ and $u(a_1, a_2) \leq 0$, so $u(a_1, a_2) = 0$, so $\overline{a_1} = \overline{a_2}$. For reflexivity, since $u(a_1, a_2) = 0$ if $\overline{a_1} = \overline{a_2}$, we must have $\overline{a_1} \geq \overline{a_2}$. For transitivity, if $\overline{a_1} \geq \overline{a_2}$ and $\overline{a_2} \geq \overline{a_3}$, then by definition we directly have $u(a_1, a_3) \geq 0$, so $\overline{a_1} \geq \overline{a_3}$. ☐

With proposition 6, the necessary and sufficient condition for weak transitive game to possess a pure-strategy Nash equilibrium can be obtained the same way as the proof of theorem 5.

Theorem 7. *A weak transitive game (A, u) possesses a pure-strategy Nash equilibrium if and only if there exists a maximal element in $\{\overline{a} \mid a \in A\}$ under the $\geq$ relation defined in 8, i.e. $\exists\, \overline{a^*} \in \{\overline{a} \mid a \in A\}$, such that $\forall\, a \in A$, $\overline{a^*} \geq \overline{a}$.*

6.2 Weak transitive game and ordinal potential game

Though weak transitive game preserves many properties from ordinal transitive game, the relationship between weak transitive game and ordinal potential game is different from that of ordinal transitive game and ordinal potential game.

Theorem 8. *In two-player symmetric zero-sum games, every ordinal potential game is a weak transitive game.*

Proof. I prove this by contradiction. Suppose (A, u) is not a weak transitive game, but is an ordinal potential game. Let P denote the potential function. By definition, there's $a_1, a_2, a_3 \in A$, such that $u(a_1, a_2) \geq 0$ and $u(a_2, a_3) \geq 0$ but $u(a_1, a_3) < 0$. Then we have $u(a_1, a_3) < u(a_2, a_3)$, therefore:

$$P(a_1, a_3) - P(a_2, a_3) < 0 \tag{3}$$

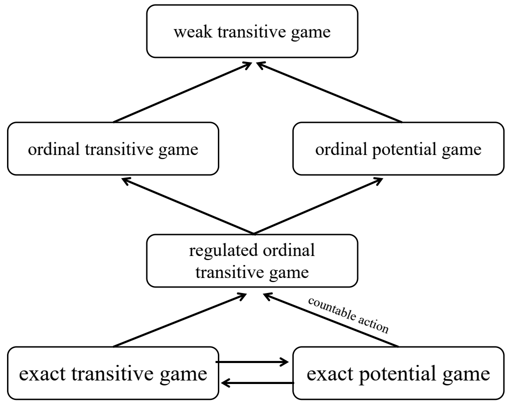

Figure 1: The hierarchy for transitive game and potential game, the directed edge represents 'is a' relationships between different class of games

Also since $u(a_2, a_1) \leq 0 \leq u(a_2, a_3)$, we have:

$$P(a_2, a_3) - P(a_2, a_1) \leq 0 \tag{4}$$

Also we have $u(a_2, a_1) \leq 0 = u(a_1, a_1)$, we have:

$$P(a_2, a_1) - P(a_1, a_1) \leq 0 \tag{5}$$

Moreover, since $u(a_1, a_1) = 0 > u(a_1, a_3)$, we have:

$$P(a_1, a_1) - P(a_1, a_3) < 0 \tag{6}$$

Summing (3), (4), (5) and (6) up, we can get $P(a_1, a_3) < P(a_1, a_3)$, a contradiction. □

However, weak transitive game and ordinal potential game are not equivalent.

Proposition 7. *Weak transitive game (A, u) is not necessarily an ordinal potential game*

Proof. Consider another two-player symmetric zero-sum game (A, u) such that $A = \{a_1, a_2, a_3\}$ and u is defined in the following matrix:

$$\begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix} \end{array}$$

the first player chooses a row and the second player chooses a column.

Through case-by-case checking, we can see (A, u) is a weak transitive game, however it is not an ordinal potential game, since $\{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_2, a_2), (a_1, a_2)\}$ is a weak improvement cycle. □

Now, I've constructed a hierarchy for transitive game and potential game, as indicated in figure 1

7 Conclusion

In this work, I studied the transitivity in strategies among two-player symmetric zero-sum games. The mathematics used in this work are quite simple, and many questions, such as what's the relationship between regulated ordinal transitive game and ordinal potential game when the action set is not countable, are left unexplored. I hope this work would provide some insights to the transitivity of strategies, and provoke more future work dedicated in this direction.

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